

A Note on Full and Complete Binary Planar Graphs

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Abstract

Let $G = (V(G), E(G))$ be a connected graph where $V(G)$ is a finite nonempty set called vertex-set of G , and $E(G)$ is a set of unordered pairs $\{u, v\}$ of distinct elements from $V(G)$ called the edge-set of G . If G is a connected acyclic graph or a connected graph with no cycles, then it is called a tree graph. A binary tree T_l with l levels is complete if all levels except possibly the last are completely full, and the last level has all its nodes to the left side. If we form a path on each level of a full and complete binary tree, then the graph is now called a full and complete binary planar graph, and it is denoted as B_n , where n is the level of the graph. This paper introduced a new planar graph which is derived from binary tree graphs. In addition, a combinatorial formula for counting its vertices, faces, and edges that depends on the level of the graph was developed.

Keywords: binary tree graph, combinatorial formula, planar graph

INTRODUCTION

Graph theory has undergone tremendous development through mathematical research (Chartrand & Zhang, 2012; Guichard, 2018). It is a growing branch of discrete mathematics that deals with graphs and their properties. To date, various graph theorists contributed to the development of such a field. Some of them are Casinillo (2018), Casinillo et al. (2017), Kumar et al. (2016), and Varkey & Thomas (2017).

In this study, we need some definitions of terms and types of graphs. A *connected graph* G is a pair of set $V(G)$ and $E(G)$, where $V(G)$ is called the *vertex-set* of graph G and $E(G)$ is a set of unordered pair $\{u, v\}$, or simply uv , of distinct elements from $V(G)$ called the *edge-set* of G , where $u, v \in V(G)$. The elements of $V(G)$ are called *vertices*, and the *cardinality* of $V(G)$ is the order of graph G , and it is denoted as $|V(G)|$. The elements of $E(G)$ are called *edges*, and the cardinality $|E(G)|$ of $E(G)$ is the size of G . If $|V(G)| = 1$, then G is called a trivial graph (Chartrand & Zhang, 2012; Ore, 1962). A *walk* is a sequence u_1, u_2, \dots, u_n of graph G vertices such that $\{u_i, u_{i+1}\} \in E(G)$ for each $i = 1, 2, \dots, n$. Vertices u_1 and u_n are the endpoints

of the walk, while the vertices u_2, u_3, \dots, u_{n-1} are *internal vertices* of the walk. The length of the walk is the number of edges on the walk, i.e., the walk u_1, u_2, \dots, u_n has length $n - 1$. A *path* is a walk that does not repeat edges and does not end where it starts, i.e., $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_n$, $u_1 \neq u_n$. A path of order n and length $n - 1$ is denoted by P_n where $n \geq 1$ (Casinillo, 2018). A *cycle* is a walk that does not repeat edges and does end where it starts, i.e., $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_n \rightarrow u_1$. A cycle graph of order n and length n is denoted by C_n where $n \geq 3$. For other graph theory concepts, readers may refer to the following references (Bollobás, 1998; Casinillo, 2020a; Casinillo et al., 2017; Chartrand & Zhang, 2012; Ore, 1962).

A *tree* is a connected acyclic graph or a connected graph with no cycles. A tree with n vertices has $n - 1$ edges (Kumar et al., 2016). The edges of a tree are known as branches, and the elements of a tree are called nodes or vertices. The nodes without child nodes are called leaf nodes. A binary tree T_l with l levels is *full* if each vertex is either a leaf or possesses exactly two child vertices. A binary tree T_l with l levels is *complete* if all levels except possibly

the last are completely full, and the last level has all its nodes to the left side (Kumar et al., 2016; Varkey & Thomas, 2017). Figure 1 shows a full and complete binary tree of level 4.

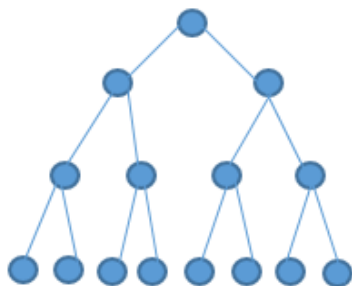


Figure 1. Graph T_4

If we form a path on each level of a full and complete binary tree, then the graph is now called a full and complete binary planar graph, and it is denoted as B_n , where n is the level of the graph. See Figure 2 below.

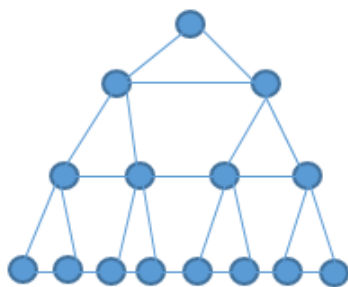


Figure 2. Graph B_4

In this paper, our effort focused on introducing a new planar graph derived from full and complete binary tree graphs. Moreover, the construction of combinatorial formula on counting its vertices, faces, and edges that depend on the level of the graph was developed.

METHOD

The methodology of this research paper is exploratory in nature which is based on the research conducted by Casinillo (2020b). This study developed a combinatorial (counting) formulae that determine the number of vertices, edges, and faces of a newly introduced planar graph $G = B_n$. These formulae were function of positive integer n , where n is the said graph level. Furthermore, the constructed formulae were characterized as n goes sufficiently large and discussed some important results.

RESULTS AND DISCUSSION

We use the following lemma below to construct the combinatorial formula for counting vertices, edges, and faces of full and complete binary planar graph B_n , where n is a positive integer.

Lemma 1. (Leithold, 1996) *Let n and r be positive integers. Then, the following holds:*

$$1 + r + r^2 + \dots + r^{n-1} = \frac{1 - r^n}{1 - r}, \text{ if } r \neq 1.$$

The following theorem is a direct result by the definition of graph B_n and Lemma 1. This theorem presents the counting formula for the order of graph $G = B_n$, where n is a positive integer.

Theorem 2: *Let $G = B_n$, where n is a positive integer. Then, $|V(G)| = 2^n - 1$.*

Proof. Suppose that $G = B_n$ where n is the level of the graph. Since every n^{th} level of the graph G contains a path of order 2^{n-1} , that is, $P_{2^{n-1}}$, then we have a series of the order of paths in G as follows:

$$1 + 2 + \dots + 2^{n-1} = \sum_{i=1}^n 2^{i-1}.$$

By Lemma 1, it implies that the order of graph G is given by

$$|V(G)| = \sum_{i=1}^n 2^{i-1} = \frac{1 - 2^n}{1 - 2} = 2^n - 1.$$

This result completes the proof. \square

Corollary 3 below is a direct consequence of Theorem 2, which determines the value of n given the order of graph $G = B_n$.

Corollary 3. *Let $G = B_n$, where n is a positive integer. Then, $n = \log_2[V(G) + 1]$.*

Proof. Obvious from Theorem 2. \square

Next, Theorem 4 presents the formula of counting the inside faces of graph $G = B_n$ for $n > 1$, and this is denoted as $F(G)$, while $F_o(G)$ denotes the outside face of graph G . Note that

for any graph $G = B_n$, we have $F_0(G) = 1$ for all values of positive integer n .

Theorem 4. Let $G = B_n$ where n is a positive integer. If $n > 1$, then $|F(G)| = 2^n - n - 1$.

Proof. We suppose that $G = B_n$ where n is a positive integer greater than 1. Now, for every n^{th} level of graph G , it contains a $2^{n-1} - 1$ number of faces. Hence, we obtained the following series for the number of faces for each n^{th} level:

$$1 + 3 + 7 + \dots + 2^{n-1} - 1 = \sum_{i=2}^n 2^{i-1} - 1.$$

Applying algebra and the concept of Lemma 1 will directly follow that the number of inside faces of graph G is given by

$$\begin{aligned} |F(G)| &= -n + \sum_{i=0}^{n-1} 2^i \\ &= -n + \frac{1 - 2^n}{1 - 2} \\ &= 2^n - n - 1. \end{aligned}$$

This result completes the proof. \square

We need the following theorem from calculus below for our next results.

Theorem 5. (L' Hôpital's Rule) (Leithold, 1996). Let $f(x)$ and $g(x)$ be differentiable functions on an open interval I , except possibly at the number a in I . Suppose that for all $x \neq a$ in I and $g(x) \neq 0$. If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ results in the indeterminate form, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists or infinite.

The next remark is a direct consequence of Theorem 2, 4, and 5.

Remark 6. Let $G = B_n$. If n is sufficiently large, then $|V(G)| = |F(G)|$.

Proof. First, we assume that n is a continuous number. Then, take the limit of the ratio of

$|V(G)|$ and $|F(G)|$ as n approaches infinity, that is,

$$\lim_{n \rightarrow \infty} \frac{|V(G)|}{|F(G)|} = \lim_{n \rightarrow \infty} \frac{2^n - 1}{2^n - n - 1}.$$

By Theorem 5, we obtained the following result,

$$\lim_{n \rightarrow \infty} \frac{|V(G)|}{|F(G)|} = 1.$$

Hence, it follows that $|V(G)| = |F(G)|$ for large values of n . It completes the proof. \square

It is worth noting that there are two kinds of the inside face of graph $G = B_n$. Those are the cycle graphs of orders 3 and 4, i.e., C_3 and C_4 , respectively. The number of C_3 and C_4 in graph $G = B_n$ are denoted as $F_3(G)$ and $F_4(G)$, respectively.

The following corollary and remark are the direct consequence of Theorem 4 and from the definition of two kinds of the inside face of graph $G = B_n$.

Corollary 7. Let $G = B_n$ where n is a positive integer. If $n > 1$, then the following holds true:

- i. $|F_3(G)| = 2^{n-1} - 1$; and
- ii. $|F_4(G)| = 2^{n-1} - n$.

Proof. To prove this corollary, we consider the two following cases:

Case 1. Suppose that if $G = B_n$, then every vertex in $(n - 1)^{\text{th}}$ possesses two child vertices for $n > 1$. Since G forms a path for every n^{th} level where $n > 1$, it implies that in every n^{th} level where $n > 1$, graph G has 2^{n-1} triangular faces. So, it implies that graph G has the series of faces as follows:

$$1 + 2 + 2^2 + \dots + 2^{n-2} = \sum_{i=2}^n 2^{i-2}.$$

Moreover, by Lemma 1, we obtained the following number of the triangular face:

$$|F_3(G)| = \sum_{i=1}^{n-1} 2^{i-1} = \frac{1 - 2^{n-1}}{1 - 2} = 2^{n-1} - 1.$$

Case 2. Since $|F(G)| = |F_3(G)| + |F_4(G)|$, then $|F_4(G)| = |F(G)| - |F_3(G)|$. Clearly, it

follows that $|F_4(G)| = 2^{n-1} - n$. This completes the proof. \square

The degree of a face of a planar graph denoted by $d(F)$ is the number of edges bounding the face F . The following remark counts the degree of the outside face of any graph $G = B_n$.

Remark 8. Let $G = B_n$ where $n > 1$. Then, $d(F_o(G)) = 2^{n-1} + 2n - 3$.

Proof. The proof is obvious. \square

The next corollary is a direct consequence of Theorem 4 and Corollary 7.

Corollary 9. Let $G = B_n$ where $n > 1$. If n is sufficiently large, then the following holds true:

- i. $|F(G)| = 2|F_3(G)|$; and
- ii. $|F(G)| = 2|F_4(G)|$.

Proof. We assume that n is a continuous number. Then, consider the following cases below:

Case 1. We take the limit of the ratio of $|F_3(G)|$ and $|F(G)|$ as n approaches infinity, that is,

$$\lim_{n \rightarrow \infty} \frac{|F_3(G)|}{|F(G)|} = \lim_{n \rightarrow \infty} \frac{2^{n-1} - 1}{2^n - n - 1}.$$

By applying the *L' Hopital's rule* successively, we end up with,

$$\lim_{n \rightarrow \infty} \frac{|F_3(G)|}{|F(G)|} = \frac{1}{2}.$$

Hence, it clearly follows that

$$|F(G)| = 2|F_3(G)|.$$

Case 2. In the same method, we take the limit of the ratio of $F_4(G)$ and $F(G)$ as n approaches infinity, and it follows that

$$\lim_{n \rightarrow \infty} \frac{|F_4(G)|}{|F(G)|} = \lim_{n \rightarrow \infty} \frac{2^{n-1} - n}{2^n - n - 1}.$$

Again, applying the *L' Hopital's rule* successively, we have

$$\lim_{n \rightarrow \infty} \frac{|F_4(G)|}{|F(G)|} = \frac{1}{2}.$$

Thus, we obtain

$$|F(G)| = 2|F_4(G)|.$$

Furthermore, this completes the proof. \square

However, for finite values of n , we obtained the following remark.

Remark 10. Let $G = B_n$. For all values of n , $|F_4(G)| < |F_3(G)|$.

Proof. The proof is obvious. \square

Theorem 11. Let $G = B_n$ where n is a positive integer. If $n > 1$, then $|E(G)| = 2^{n+1} - n - 3$.

Proof. If we suppose that $G = B_n$ where n is a positive integer greater than 1, then for every n^{th} level of the graph G , it contains $2^n - 1$ edges. Thus, it follows that we obtained the series for the number of edges,

$$3 + 7 + \dots + (2^n - 1) = \sum_{i=2}^n 2^i - 1.$$

Applying algebra and the concept of Lemma 1, it clearly follows that

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{n-1} 2^{i+1} - 1 \\ &= 4 \left[\frac{1 - 2^{n-1}}{1 - 2} \right] - (n - 1). \end{aligned}$$

Hence, $|E(G)| = 2^{n+1} - n - 3$.

This completes the proof. \square

The corollary that follows is a direct consequence of Theorem 2, 4, and 11.

Corollary 12. Let $G = B_n$ where $n > 1$. If n is sufficiently large, then the following holds true:

- i. $|E(G)| = 2|V(G)|$; and
- ii. $|E(G)| = 2|F(G)|$.

Proof. The proof is similar to the proof of Corollary 9. \square

Lemma 13. (Euler's formula). If G is a connected planar graph, then

$$|V(G)| + |F(G)| = |E(G)| + 2.$$

Because of Euler's formula above, the following remark is obtained.

Remark 14. Graph $G = B_n$ is a planar graph that satisfies the Euler's formula.

Proof. The proof is obvious. \square

Lemma 15. (Chartrand & Zhang, 2012) For any simple connected planar graph G , we have

$$\sum_i d(F_i) = 2|E(G)|.$$

Considering Lemma 15, the following remark is a direct consequence showing that graph $G = B_n$ is a planar graph.

Remark 16. Let $G = B_n$ where $n > 1$. Then,
 $d(F_0(G)) + 3|F_3(G)| + 4|F_4(G)| = 2|E(G)|$.

Proof. The proof is obvious. \square

CONCLUSION

This paper introduced a new planar graph that is based on a full and complete binary tree graph. A combinatorial formula was developed to count its vertices and found out that there are $2^n - 1$ vertices, where n is the level of the graph. Results showed three kinds of faces in graph $G = B_n$, that is, outside face, C_3 face, and C_4 face denoted as F_0 , F_3 , and F_4 , respectively. The study also developed counting formulae for the different kinds of faces in the graph. Furthermore, the number of edges in graph $G = B_n$ was also determined by developed counting formula that also depends on the level of the graph. It was also found out that the three parameters, such as vertices, faces, and edges of graph $G = B_n$ satisfy Euler's formula for planar graphs. Lastly, the paper had discussed some properties regarding the said three parameters as the level of graph n goes sufficiently large using L' Hôpital's Rule in calculus. Future

research may consider counting the domination number of the full and complete binary planar graph. Furthermore, it would be interesting if the graph $G = B_n$ will be characterized using graph labeling concepts.

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